

A Home Experiment in Elasticity

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Abstract

We analyze a simple problem in elasticity: the *initial* motion of an elastic bar that after being hanged from an end is suddenly released. In a second problem a point mass is attached on the top of the bar. The analytical solutions uncover some unexpected properties, which can be checked, with a digital camera or camcorder, in an alternative setup in which a spring is substituted for the bar. The theoretical model and the experiments are useful to understand the similarities and differences between the elastic properties of bar and spring. Students can take advantage of the home experiments to improve their understanding of elastic waves.

I. INTRODUCTION

“A thin long elastic bar is vertically suspended from one end by means of a string.

After the equilibrium is attained, the string is suddenly cut. Will all bar points have the same acceleration *just after the string is cut?* —asks the teacher.

Yes sir —answers Peter— according to the equivalence principle all particles will fall with the same acceleration in an homogeneous gravitational field.

But I have read in the solution to problem 79 of Bújovtsev’s book¹ that although the lower end will have acceleration g the upper end will fall faster —says Ann.

Of course —stresses Mary—, when we computed the tension in the equilibrium state we saw that the tension is null in the lower end, so that only gravity will exert a force on particles there, while the tension is maximum in the upper end, which will in consequence have higher acceleration.

What do you think?”

When, some weeks ago, one of us proposed the previous question to our students we did not suspect the full interest of the problem. Obviously the answers in the quotation are all wrong. The equivalence principle only applies to free falling particles and in the problem there are also elastic forces. On the other hand, g will be the acceleration of the center of mass: so it is not possible to have an acceleration equal to g at the lower end and greater values everywhere else, as Ann and Mary think. Our students found the problem highly unintuitive. It took some time to convince them that although the tension in the upper end disappears when the string is cut, it will pass some time before the tension and, as a consequence of Hooke’s law, the deformation change in other points. The change of state will propagate along the bar in the form of an elastic wave, so that *initially* the lower end does not move at all.

To help them understand the phenomenon we devised a simple experiment. We will show below that the theoretical analysis in the case of a metallic bar is elementary, but the deformation would be too small and change too fast to be seen, except maybe with some sophisticated experimental setup. Instead we used a plastic spring², which when stretched behaves much like the elastic bar, but have elastic properties that change completely when the loops are in contact with each other. In the spring deformations are large and change slowly enough to be recorded with a standard digital camera, since most of them have the ability to record short video sequences. Although the result of our first try³ was not of good quality, one could easily see that the lower end did not start

falling until the deformation change reached it.

One could then think that if the center of mass moves with acceleration g while the points at the lower end are still at rest, the upper points must move with high accelerations. We will see in the following that this is not the case with the bar: what happens is something simpler but (probably) less intuitive. In fact, one can go beyond the qualitative analysis and compute the full evolution (from the string being cut to the deformation change reaching the lower end) with a minimal mathematical apparatus, by using concepts available to students of introductory physics courses. The analysis of Section II predicts that after the deformation change has reached a point it will start moving with a velocity independent of position and time, i.e., without acceleration. This unexpected result is a consequence of having assumed that the force exerted by the string vanishes instantaneously, but one could expect it to be a good approximation in appropriate cases. To check the theoretical prediction we went back to our video, extracted some consecutive frames with a free tool⁴ and get something like Figure 1, where one can see that the upper end moves with a more or less constant velocity, while the lower end remains at rest until the elastic wave reaches this point.

However, as discussed in Section III, except for the first frames in Figure 1, there was not good agreement with the theoretical analysis because the upper coils quickly became completely compressed and touched one another. The reason can be understood by using the theoretical model: matter quickly moves faster than the elastic wave and the dynamical problem changes completely (this new problem was discussed by Calkin⁵).

We then realized that we could get better agreement between the theory of elastic waves and the experiment with the spring by modifying the problem by attaching a mass on top of bar and spring. As described in Section IV, however small the mass is, some properties of the solution for the bar change qualitatively, and not only quantitatively. This in turn allows to apply for longer spans of time the theoretical model for the bar to the experimental setup with the spring.

II. THE FALLING BAR

In the following we will label each point P with the distance x measured from end A when the bar is not strained, as shown in Figure 2 a).

A. Initial equilibrium

For times $t < 0$ the bar is hung as shown in Figure 2 b), so that the distance AP is now $x + u_0(x)$ in terms of the deformation field u_0 . The tension τ_0 is readily computed by writing the equilibrium condition of PB and using the fact that tension vanishes at B : $\tau_0(L) = 0$. One gets

$$\tau_0(x) = \rho g(L - x), \quad (1)$$

where ρ is the mass density. Hooke's law and the boundary condition $u_0(0) = 0$ allow computing the deformation:

$$\tau_0 = E \frac{\partial u_0}{\partial x} \implies u_0(x) = \frac{g}{2c^2} (2Lx - x^2), \quad (2)$$

where E is Young's modulus and $c \equiv \sqrt{E/\rho}$ the speed of sound in the bar. We are assuming here the unitary deformation $\partial u_0/\partial x$ is small enough to satisfy Hooke's law, which implies gL/c^2 is small.

B. Initial dynamics

At $t = 0$ the string is cut so that the tension at A disappears instantaneously, and a discontinuity starts propagating along the bar with velocity c . It is easy to use Newton's second law and Hooke's law to find the wave equation satisfied by the deformation field $u(t, x)$, which gives the position of P at any time, as depicted in Figure 2 c):

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + g. \quad (3)$$

However, probably it will be easier for students to understand the calculation in a reference frame falling with the center of mass, where the deformation field will be

$$u^* = u - \frac{1}{2}gt^2. \quad (4)$$

In such a frame the inertial force and weight cancel each other and the remaining forces are of elastic origin, so that the longitudinal evolution equation is the homogeneous wave equation introduced in elementary physics courses⁶,

$$\frac{\partial^2 u^*}{\partial t^2} = c^2 \frac{\partial^2 u^*}{\partial x^2}, \quad (5)$$

whose solution is early taught to students in d'Alembert's form, i.e., as a superposition of two waves propagating in opposite directions⁶:

$$u^*(t, x) = f^*(x - ct) + h^*(x + ct). \quad (6)$$

(Of course, this is the standard way of solving inhomogeneous linear equations, but the reasoning here is more physical and can be presented to students before they study differential equations.)

For $x > ct$ the perturbation has not reached the point, so that we have $u^* = u_0^*$. In consequence we seek a piecewise solution as follows:

$$u^*(t, x) = \begin{cases} u_0(x) - \frac{1}{2}gt^2, & x > ct; \\ f(x - ct) + h(x + ct), & x < ct < L. \end{cases} \quad (7)$$

For $x < ct$ we have a solution of (5) in terms of two functions, f and h , which is easily computed by using the following two physical conditions:

- The bar does not break so that u is continuous at the wavefront $x = ct$:

$$-\frac{g}{4c^2}2x(2x - 2L) = f(0) + h(2x) \implies h(x) = \frac{g}{4c^2}x(2L - x) - f(0). \quad (8)$$

Since only the combination $f(x - ct) + h(x + ct)$ will appear in the solution, there is no restriction in taking $f(0) = 0$.

- In this reference frame the center of mass is at rest, so that its velocity is:

$$\begin{aligned} 0 &= \frac{1}{L} \int_0^L \frac{\partial u^*(t, x)}{\partial t} dx \\ &= \frac{1}{L} \int_{ct}^L (-gt) dx + \frac{c}{L} \int_0^{ct} [h'(x + ct) - f'(x - ct)] dx \\ &= \frac{c}{L} f(-ct) - \frac{g}{4L} t(2L - ct). \end{aligned} \quad (9)$$

In consequence we have

$$f(x) = -\frac{g}{4c^2}x(2L + x). \quad (10)$$

Remembering (4), we get our main result in the laboratory frame:

$$u(t, x) = \frac{g}{2c^2} \begin{cases} 2Lx - x^2, & x > ct; \\ 2Lct - x^2, & x < ct < L. \end{cases} \quad (11)$$

C. Properties of the solution

The stress is then

$$\tau(t, x) = E \frac{\partial u}{\partial x} = \rho g \begin{cases} L - x, & x > ct; \\ -x, & x < ct < L. \end{cases} \quad (12)$$

We have, as expected, $\tau(t, 0) = \tau(t, L) = 0$ for all $0 < t < L/c$. On the other hand, the tension is discontinuous and becomes compression at the wavefront, and solution (12) at the lower end $x = L$ goes to $-\rho g L$ as $t \rightarrow L/c$. This proves there must appear a reflected wave to make sure $\tau(t, L) = 0$ always; but we are not interested here in this reflected wave, for it cannot be seen in our home experiment.

The surprise arises when one computes the velocity of each point:

$$\frac{\partial u}{\partial t} = \frac{gL}{c} \begin{cases} 0, & x > ct; \\ 1, & x < ct < L. \end{cases} \quad (13)$$

All points outside the wavefront move without acceleration; but more and more points quit rest and start moving with velocity gL/c (which, as stressed after (2), is smaller than c) as time increases, so that the center of mass moves with increasing velocity gt . Put in other way, at the wavefront $x = ct$ the velocity is discontinuous and, in consequence, the acceleration infinite. For students knowing Dirac's delta function —which can be easily introduced to physics students⁷— one can write the acceleration in the following form:

$$\frac{\partial^2 u}{\partial t^2} = gL \delta(x - ct), \quad (t < L/c). \quad (14)$$

We see now that the answer to the question proposed at the beginning of the introduction is that, in the limit in which the string is cut instantaneously, initially all points move without acceleration, except for those points lying at the wavefront, which have infinite acceleration. This problem illustrates a rather unusual way for a system of particles to gain more and more linear momentum under an external force: it is enough to keep putting particles in motion, although all velocities are equal and, more strikingly, constant.

III. THE FIRST EXPERIMENT

The theoretical analysis of the problem with an elastic bar can be carried out with elementary physics; in fact, it is enough to use the first example of wave equation discussed in some

textbooks⁸. In our home experiment we needed bigger deformations and slower propagation velocities, so we considered using instead a spring. The complete study of the latter is more difficult: deformations are no longer small, the elastic properties when stretched and under compression are completely different, when hung from an end it stretches but also develops non negligible torsion which changes when moving, transversal motion is likely to appear, and so on. However, one could expect that at least some qualitative results (such as the points starting moving progressively with constant velocity) would be the same as in the elastic bar.

Instead of an elastic bar we released a colorful plastic spring² with some black tags stuck every third loop. We used a digital camera⁹ to shoot a short video sequence at 30 frames per second. The resulting animation is displayed (at two different speeds) in a web page¹⁰. One can clearly see there that the tags and the lower end remain at rest for a while. To further explore the process, we used a tool⁴ to extract some consecutive frames, which are displayed in Figure 1.

At first sight one might conclude that the tagged points start moving all with the same constant velocity only when the elastic wave reaches them, but a simple calculation shows disagreement between theory and experiment: the upper coils quickly become completely compressed. Initially the solution computed for the bar is approximately valid for the spring —and was computed in this context, by means of a somewhat less elementary mathematical method by Calkin⁵ (see also Cushing¹¹)—: one only has to replace c by $L\sqrt{k/m}$, where m and k are respectively the mass and the elastic constant of the spring. But, unlike in a metallic bar, in a soft spring the velocity gL/c of the coils above the frontwave $x = ct$ quickly become bigger than the velocity of the latter:

$$\frac{d}{dt} [ct + u(t, ct)] = \frac{gL}{c} + c - gt. \quad (15)$$

From $t = c/g$ on (in fact, a bit earlier due to the finite thickness of the coils) an increasing number of upper coils touch one another and fall over the coils below before there is time for the tension to change there. We have then a kind of matter wave that moves faster than the elastic wave created when the spring was released. The dynamical problem is then completely different and was analyzed by Calkin⁵. One can check his solution in our case ($l_0 \approx l_1 \approx 6.5$ cm, $\xi_1 \approx 1$, $k/m \approx 4$ s⁻², in his notation): good numerical agreement is shown in Figure 3. This may be an interesting problem in dynamics of systems of particles, but we were more interested in elasticity, so that we turned to the study of the problem in Section IV.

We can see that the solution (11) breaks at $t = c/g$ (provided it is less than L/c , which would

never happen for an actual metallic bar) from another point of view. At that moment

$$\frac{\partial}{\partial t} [x + u(t, x)] = 1 - \frac{gx}{c^2} \quad (16)$$

becomes negative at the wavefront $x = ct$, which is clearly impossible, for it would mean an inversion of the spatial order of coils.

IV. THE FALLING BAR WITH AN ATTACHED POINTLIKE MASS

Let us consider again the bar of Figure 2 but let us assume in the upper end A there is attached a pointlike mass $M = \mu m$. The analysis of Section II only changes from (9) on. In the latter we now have to take into account the contribution from M :

$$\frac{1}{L} \int_0^L \frac{\partial u^*(t, x)}{\partial t} dx + \mu \frac{\partial u^*}{\partial t}(t, 0) = 0, \quad (17)$$

which leads to the condition

$$\mu L f'(x) - f(x) = \frac{g}{4c^2} [2\mu L^2 + 2(1 + \mu)Lx + x^2]. \quad (18)$$

This is a first-order linear equation with constant coefficients which can be solved, along with the initial condition $f(0) = 0$, in a number of easy ways (integrating with respect to x after multiplying it with $e^{-x/\mu L}$ or by using computer algebra, for instance) to give

$$f(x) = -\frac{g}{4c^2} [x(2L + x) - 2q_\mu(x)], \quad q_\mu(x) \equiv 2\mu L [(1 + \mu)L(e^{x/\mu L} - 1) - x]. \quad (19)$$

In consequence, the deformation field in the laboratory frame is

$$u(t, x) = \frac{g}{2c^2} \begin{cases} 2Lx - x^2, & x > ct; \\ 2Lct - x^2 + q_\mu(x - ct), & x < ct < L, \end{cases} \quad (20)$$

which reduces to (11) in the limit $\mu \rightarrow 0$, since $\lim_{\mu \rightarrow 0} q_\mu(x - ct) = 0$ for $x < ct$. The velocity

$$\frac{\partial u}{\partial t} = \frac{g(1 + \mu)L}{c} (1 - e^{(x-ct)/\mu L}) \begin{cases} 0, & x > ct; \\ 1, & x < ct < L \end{cases} \quad (21)$$

is now, unlike in (13), continuous through the wavefront $x = ct$ for any mass ratio $\mu > 0$.

This qualitatively different behavior may seem at first counterintuitive, but there is a clear physical reason for it: now the stress

$$\tau(t, x) = \rho g \begin{cases} L - x, & x > ct; \\ -\mu L - x + (1 + \mu)L e^{(x-ct)/\mu L}, & x < ct < L \end{cases} \quad (22)$$

does not becomes instantaneously null at $t = 0$ and $x = 0$, but retains its previous value, $\tau(0, 0) = \rho g L$, because of the attached mass, however small it is. For the same reason the points above the frontwave are now accelerated:

$$\frac{\partial^2 u}{\partial t^2} = g \frac{1 + \mu}{\mu} e^{(x-ct)/\mu L}, \quad \text{for } x < ct < L. \quad (23)$$

Although now the velocity at the wavefront is zero, the solution would break at the moment

$$t = \frac{\mu L}{c} \log \frac{(1 + \mu)L}{\mu L + x - c^2/g} + \frac{x}{c}, \quad (x < ct < L), \quad (24)$$

provided it is real and less than L/c , as one can see by repeating the calculation in (16). One can check that in the case of our spring this value increases with μ , so that one would expect our analytical solution (20) to be valid for longer intervals with heavier masses.

Notice that when the elastic wave reaches a point its acceleration is $g(1 + \mu)/\mu > g$, and the same happens initially at the upper end A .

V. THE SECOND AND THIRD EXPERIMENTS

This can be clearly seen in our second experiment¹² where a thin wooden slab is fixed at the top end A . A thicker block is then put on top of it. When the spring is released, the acceleration of A is greater than that of the block, which immediately splits up and follows the familiar free fall trajectory, as displayed in the consecutive frames of Figure 4. One can see there that the lightweight mass ($\mu \approx 0.22$) is enough to make sure the upper coils are also stretched for a while and, in consequence, to avoid our continuous theoretical model break early because of the finite thickness of the coils and the fact that the elastic properties of the spring change completely when the coils touch one another.

This can be seen even better in our third experiment¹³, where the thin slab is replaced by a thicker block ($\mu \approx 2.35$). One can clearly see in Figure 5 that each black tag starts moving only when the stretching begins to change there, i.e., when the elastic waves reaches it. In the same figure we have plotted the trajectories of the top most coil and the first three tags as computed with solution (20): one gets good agreement, despite the amateurish experiment and the differences between bar and spring.

VI. CONCLUSIONS

We have analyzed, by using elementary physical concepts, a couple of problems in elasticity with results difficult to anticipate by pure intuition. To help students understand elastic waves one can easily check the most striking aspects of the analytical results in an alternative setup, in which the elastic bar is substituted by a spring, which allows far bigger deformations and slower wave propagation. The similarities and differences between bar and spring can be used in an illustrative discussion in introductory physics courses.

Since one only needs a digital camera (or a video camera) and some freeware to process the video sequences, the experiments can be easily performed in the classroom and repeated at home by the interested student.

One can take advantage of the widespread availability of digital cameras to visually check the solution of other problems in mechanics. Let us only mention a well-known one: the free end of an articulated arm released from an angle below $\arcsin 1/\sqrt{3} \approx 35^\circ$ is always accelerated faster than g ¹⁴.

Acknowledgments

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¹ B. B. Bújovtsev, V. D. Krívchenkov, G. Ya. Miákishev and I. M. Saráeva, *Problemas seleccionados de física elemental*, Mir, Moscow (1979).

² “Rainbow Magic Spring” from www.physlink.com.

³ See <http://tp.lc.ehu.es/jma/mechanika/jarraitua/slinky.html>.

⁴ VirtualDub from <http://www.virtualdub.org/>.

⁵ M. G. Calkin, “Motion of a falling spring,” *Am. J. Phys.* **61**, 261–264 (1993).

⁶ M. Alonso and E. J. Finn, *Physics*, Addison-Wesley, Reading, Mass., (1992), § 28.4.

⁷ J. M. Aguirregabiria, A. Hernández and M. Rivas, “ δ -function converging sequences,” *Am. J. Phys.* **70**, 180–185 (2002).

⁸ M. Alonso and E. J. Finn, *Physics*, Addison-Wesley, Reading, Mass., (1992), § 28.5.

⁹ Canon PowerShot S1 IS.

¹⁰ See <http://tp.lc.ehu.es/jma/mekanika/jarraitua/spring.html>.

¹¹ J. T. Cushing, “The method of characteristics applied to the massive spring problem,” *Am. J. Phys.* **52**, 933–937 (1984).

¹² See <http://tp.lc.ehu.es/jma/mekanika/jarraitua/spring2.html>.

¹³ See <http://tp.lc.ehu.es/jma/mekanika/jarraitua/spring3.html>.

¹⁴ See, for instance, <http://tp.lc.ehu.es/jma/mekanika/solidoa/fasterg.html>.

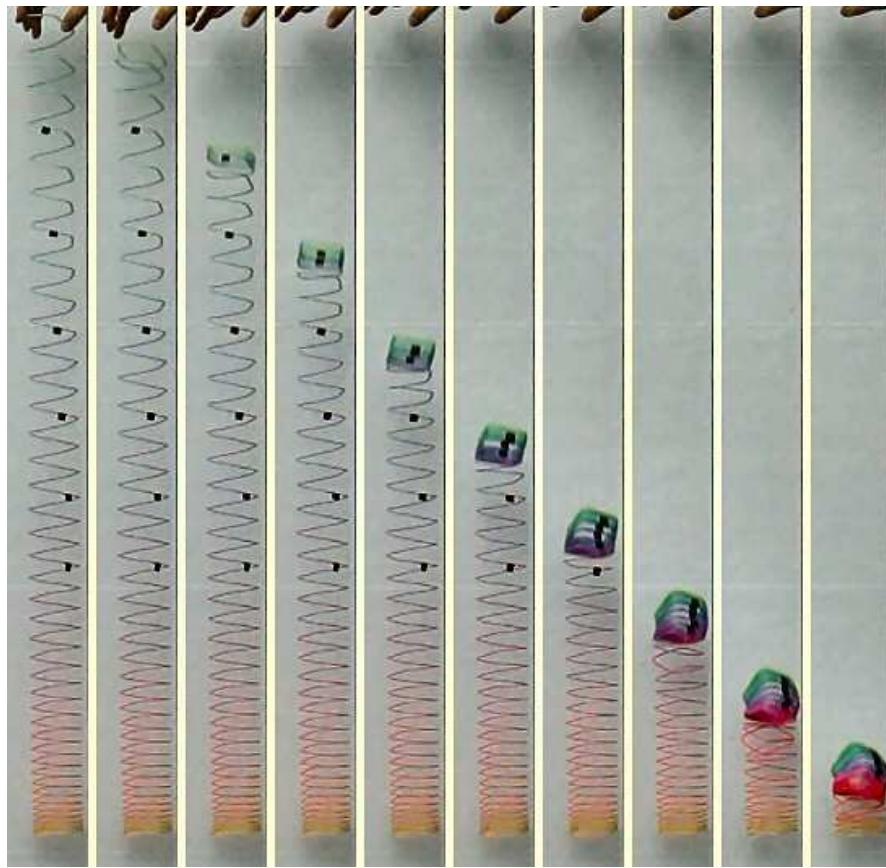


FIG. 1: Some consecutive frames from the video sequence¹⁰.

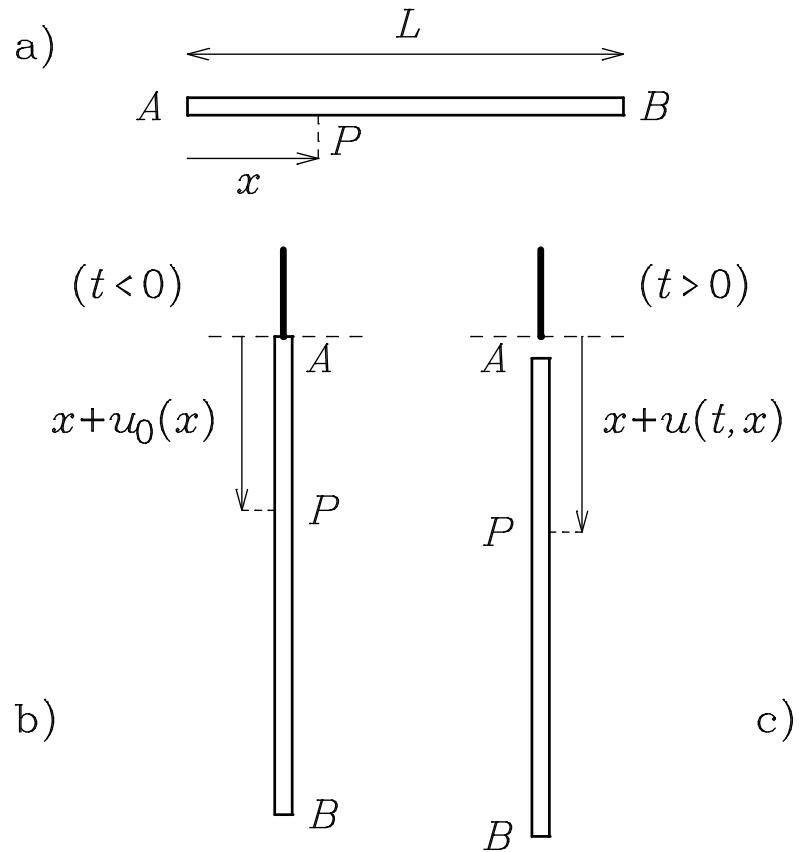


FIG. 2: Elastic bar a) without strain, b) hanging and c) released.

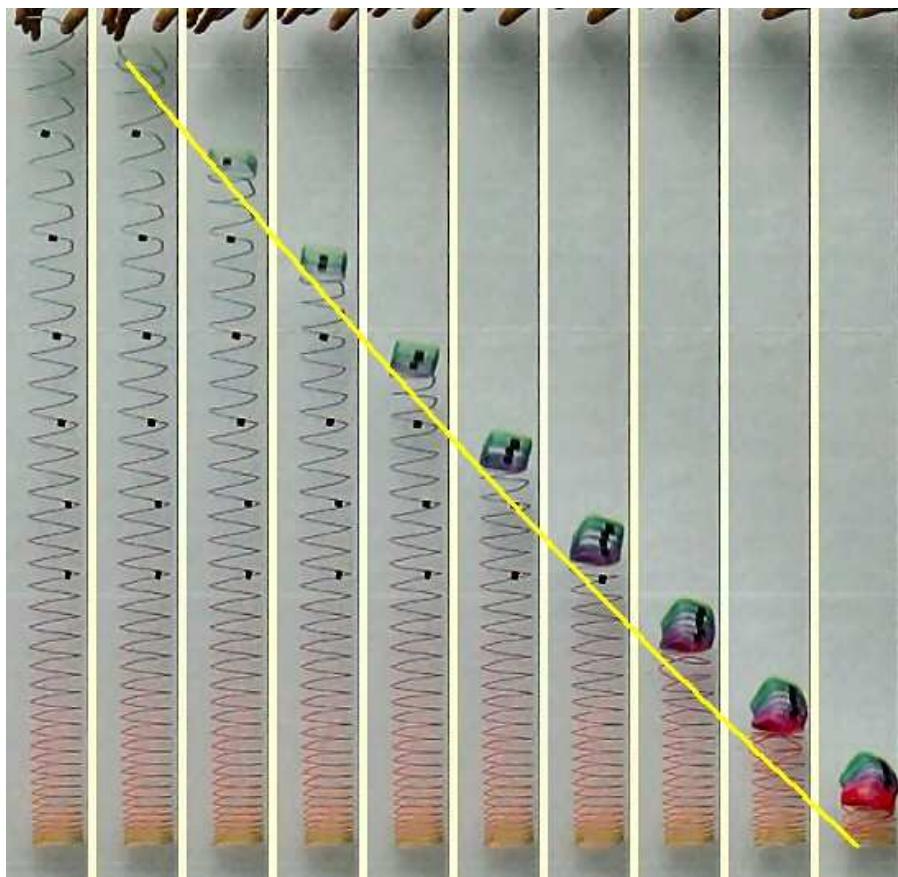


FIG. 3: Same as Figure 1 along with the solution by Calkin⁵.

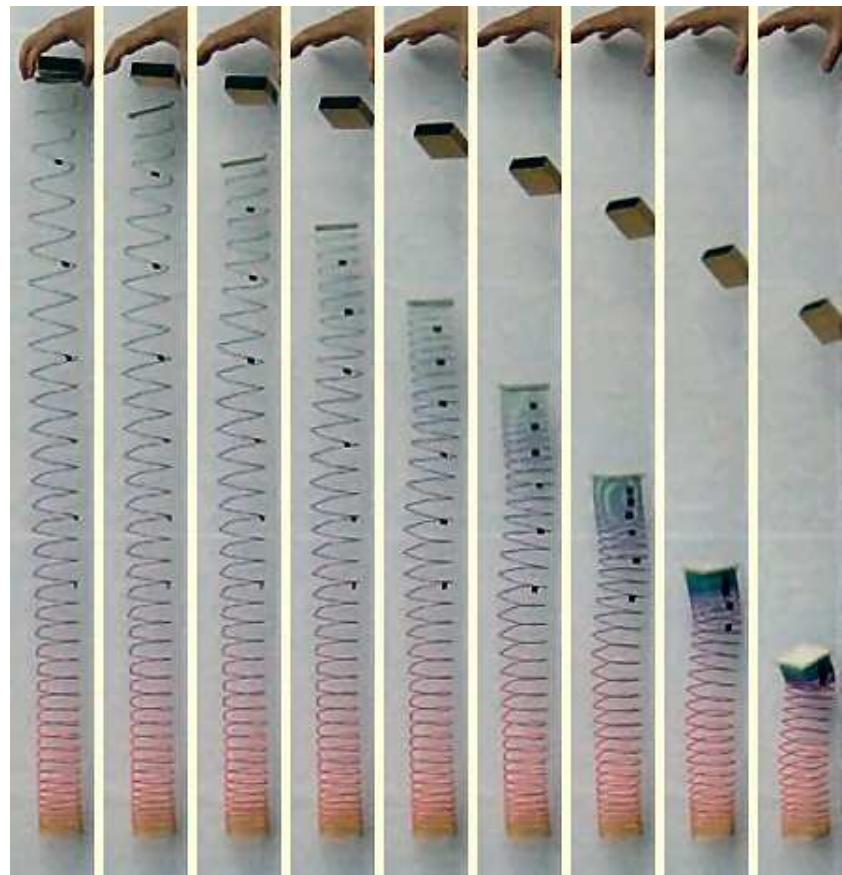


FIG. 4: Same as Figure 1 with a thin slab fixed at the upper end and a block on top of it.

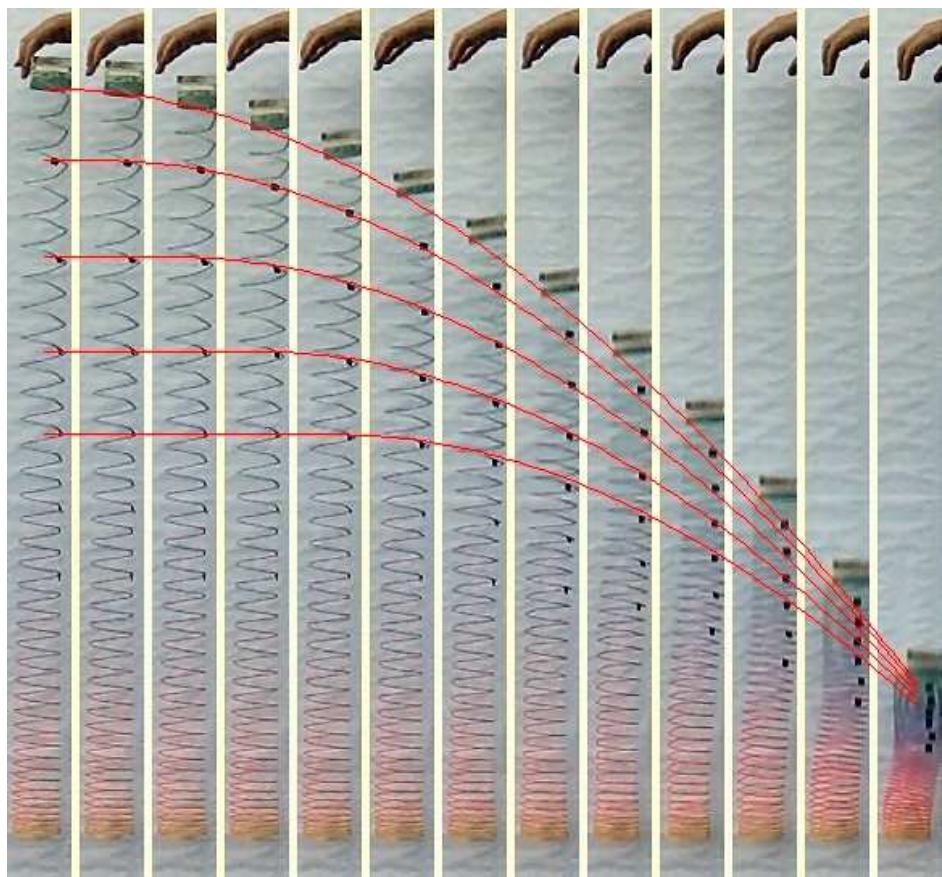


FIG. 5: Same as Figure 1 with a wooden block ($\mu \approx 2.35$) fixed at the top end.